ORBITS OF PARABOLIC SUBGROUPS ON METABELIAN IDEALS

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ABSTRACT. Let k be an algebraically closed field, $t \in \mathbb{Z}_{\geq 1}$, and let B be the Borel subgroup of $\mathrm{GL}_t(k)$ consisting of upper-triangular matrices. Let Q be a parabolic subgroup of $\mathrm{GL}_t(k)$ that contains B and such that the Lie algebra $\mathfrak{q}_u$ of the unipotent radical of Q is metabelian, i.e. the derived subalgebra of $\mathfrak{q}_u$ is abelian. For a dimension vector $\mathbf{d} = (d_1, \ldots, d_t) \in \mathbb{Z}_{\geq 1}^t$ with $\sum_{i=1}^t d_i = n$, we obtain a parabolic subgroup $P(\mathbf{d})$ of $\mathrm{GL}_n(k)$ from B by taking upper-triangular block matrices with (i,j) block of size $d_i \times d_j$. In a similar manner we obtain a parabolic subgroup $Q(\mathbf{d})$ of $\mathrm{GL}_n(k)$ from Q. We determine all instances when $P(\mathbf{d})$ acts on $\mathfrak{q}_u(\mathbf{d})$ with a finite number of orbits for all dimension vectors $\mathbf{d}$. Our methods use a translation of the problem into the representation theory of certain quasi-hereditary algebras. In the finite cases, we use Auslander–Reiten theory to explicitly determine the $P(\mathbf{d})$-orbits; this also allows us to determine the degenerations of $P(\mathbf{d})$-orbits.

1. INTRODUCTION

Let G be a reductive algebraic group over an algebraically closed field k, and assume that the characteristic of k is good for G. There has been a great deal of recent interest in the adjoint action of a parabolic subgroup P of G on the Lie algebra $\mathfrak{p}_u$ of its unipotent radical. Such actions were considered in the case where $\mathfrak{p}_u$ is abelian by R. Richardson, R. Steinberg and the third author in [13]; in particular, they showed that if $\mathfrak{p}_u$ is abelian, then there is always a finite number of P-orbits in $\mathfrak{p}_u$ and gave a parameterization of these orbits. Subsequently, in work of U. Jürgens and the second and third authors, a classification of all instances when P acts on $\mathfrak{p}_u$ with finitely many orbits has been obtained, see [8] and [12]. In addition, for G simple not of type E_7 or E_8 there is a classification of all instances when there are finitely many P-orbits in higher terms $\mathfrak{p}_u^{(l)}$ of the descending central series of $\mathfrak{p}_u$, see [2], [3] and [6]. For further information on parabolic group actions we refer the reader to the survey [16].

In the case $G = \mathrm{GL}_n(k)$, there has been much success in understanding the adjoint action of a parabolic subgroup through a translation in to the representation theory of certain quasi-hereditary algebras. This translation was first observed in [8], and has subsequently been further exploited, see for example, [2] and [4]. We refer the reader to [7] and [11] for recent related developments.

In this paper we consider a related problem in case $G = \mathrm{GL}_n(k)$. Rather than considering the action of P on $\mathfrak{p}_u$ (or $\mathfrak{p}_u^{(l)}$), we study the action of P on $\mathfrak{q}_u$, where Q is a parabolic subgroup of G such that $\mathfrak{q}_u$ is metabelian. Our main result, Theorem 1.1, gives a finiteness condition for such actions.

Let $G = \mathrm{GL}_n(k)$, where k is an algebraically closed field. Let $\mathbf{d} = (d_1, \ldots, d_t) \in \mathbb{Z}_{\geq 1}^t$ satisfy $\sum_{i=1}^t d_i = n$; call such a t-tuple $\mathbf{d}$ a *dimension vector*. Given a dimension vector

$\mathbf{d}$, define the parabolic subgroup $P = P(\mathbf{d})$ of G to be the stabilizer of the *standard flag* $0 \subseteq k^{e_1} \subseteq k^{e_2} \subseteq \ldots \subseteq k^{e_{t-1}} \subseteq k^{e_t} = k^n$ in k^n, where $e_i = \sum_{j=1}^{i} d_j$. Let $\mathbf{a} = (a_1, a_2, a_3) \in \mathbb{Z}^3_{\geq 1}$ with $a_1 + a_2 + a_3 = t$ and define $b_i = \sum_{j=1}^{i} a_j$ for $i = 0, 1, 2, 3$. Define $Q = Q(\mathbf{a}, \mathbf{d}) = P(e_{b_1}, e_{b_2} - e_{b_1}, e_{b_3} - e_{b_2}) \supseteq P$. Then P acts on the Lie algebra $\mathfrak{q}_u = \mathfrak{q}_u(\mathbf{a}, \mathbf{d})$ of the unipotent radical of Q via the adjoint action.

Define the partial order $\leq$ on the set of triples $\mathbf{a} \in \mathbb{Z}^3_{\geq 1}$, by $\mathbf{a} \leq \tilde{\mathbf{a}}$ if $a_i \leq \tilde{a}_i$, for $i = 1, 2, 3$. The *reverse* of a triple $\mathbf{a}$ is $\mathbf{a}_{\mathrm{rev}} = (a_3, a_2, a_1)$.

We can now state our main result which gives a classification of all triples $\mathbf{a}$ such that $P(\mathbf{d})$ acts with a finite number of orbits on $\mathfrak{q}_u(\mathbf{a}, \mathbf{d})$ for all dimension vectors $\mathbf{d}$.

Theorem 1.1. *Let* $\mathbf{a} \in \mathbb{Z}^3_{\geq 1}$ *and* $\mathbf{d} \in \mathbb{Z}^t_{\geq 1}$ *a dimension vector with* $a_1 + a_2 + a_3 = t$.
  (1) *Assume* $\mathbf{a}$ *and* $\mathbf{d}$ *are as in Table 1. Then* $P(\mathbf{d})$ *acts on* $\mathfrak{q}_u(\mathbf{a}, \mathbf{d})$ *with an infinite number of orbits.*
  (2) *Assume* $\mathbf{a}$ *is a triple in Table 2. Then* $P(\mathbf{d})$ *acts on* $\mathfrak{q}_u(\mathbf{a}, \mathbf{d})$ *with a finite number of orbits for all* $\mathbf{d}$.
  (3) *Any triple* $\mathbf{a}$ *is either: less than or equal to a triple in Table 2 or its reverse; or greater than or equal to a triple in Table 1 or its reverse. In the former case there is a finite number of* $P(\mathbf{d})$*-orbits in* $\mathfrak{q}_u(\mathbf{a}, \mathbf{d})$ *for all* $\mathbf{d}$*, and in the latter case there is an infinite number of* $P(\mathbf{d})$*-orbits in* $\mathfrak{q}_u(\mathbf{a}, \mathbf{d})$ *for some* $\mathbf{d}$*.*

$\mathbf{a}$	$\mathbf{d}$	$\dim P/Q_u$
$(1,3,5)$	$(3,2,2,2,1,1,1,1,1)$	48
$(1,4,3)$	$(2,1,1,1,1,1,1,1)$	20
$(1,6,2)$	$(3,1,1,1,1,1,1,2,2)$	42
$(2,2,5)$	$(2,2,3,3,1,1,1,1,1)$	54
$(2,3,2)$	$(1,1,1,1,1,1,1)$	12
$(3,2,3)$	$(1,1,1,2,2,1,1,1)$	24

TABLE 1. The minimal infinite triples $\mathbf{a}$

$\mathbf{a}$	$(1,2,a)$	$(1,a,1)$	$(a,1,b)$	$(1,3,4)$	$(1,5,2)$	$(2,2,4)$

TABLE 2. The maximal finite triples $\mathbf{a}$ $(a, b \in \mathbb{Z}_{\geq 1})$

Remarks 1.2. (i). Observe that $P(\mathbf{d})$ has a finite number of orbits on $\mathfrak{q}_u(\mathbf{a}, \mathbf{d})$ for all $\mathbf{d}$ if and only if the same is true for the action of $P(\mathbf{d})$ on $\mathfrak{q}_u(\mathbf{a}_{\mathrm{rev}}, \mathbf{d})$. This easy observation means that Theorem 1.1 does indeed give the desired classification of all triples $\mathbf{a}$ such that $P(\mathbf{d})$ acts on $\mathfrak{q}_u(\mathbf{a}, \mathbf{d})$ with a finite number of orbits for all $\mathbf{d}$.
  (ii). Note that Theorem 1.1 is consistent with the computer calculations made in [9].
  (iii). Compared with Theorem 1.1, it is a much harder problem to determine all pairs $(\mathbf{a}, \mathbf{d})$ such that $P(\mathbf{d})$ acts on $\mathfrak{q}_u(\mathbf{a}, \mathbf{d})$ with a finite number of orbits. For the minimal infinite cases

given in Table 1, we have that $P(\mathbf{d})$ acts on $\mathfrak{q}_u(\mathbf{a}, \mathbf{d})$ with finitely many orbits if one entry of $\mathbf{d}$ is less than that for the $\mathbf{d}$ given in the table. For larger values of $\mathbf{a}$ it can also be the case that $P(\mathbf{d})$ acts on $\mathfrak{q}_u(\mathbf{a}, \mathbf{d})$ with finitely many orbits for some values of $\mathbf{d}$. It seems infeasible to determine all such $\mathbf{d}$ for all $\mathbf{a} \in \mathbb{Z}_{\geq 1}^3$.

We prove Theorem 1.1 in the next section. The proof of (1) is a dimension counting argument that is elementary; it essentially involves calculating the dimension given in the third column of Table 1. The majority of the work required is in proving (2). To do this we interpret the problem in terms of the representation theory of a certain quasi-hereditary algebra $\mathcal{A}(\mathbf{a})$. The isoclasses of Δ-filtered $\mathcal{A}(\mathbf{a})$-modules with Δ-dimension vector $\mathbf{d}$ correspond to the orbits of $P(\mathbf{d})$ in $\mathfrak{q}_u(\mathbf{a}, \mathbf{d})$. For each of the triples in Table 2, we have calculated the Auslander–Reiten quiver of Δ-filtered $\mathcal{A}(\mathbf{a})$-modules. In doing so we see that for these values of $\mathbf{a}$ the algebra $\mathcal{A}(\mathbf{a})$ has finite Δ-representation type, which proves (2).

In order to prove (3) we just require the following observation. If $\tilde{\mathbf{a}} \leq \mathbf{a}$, then there is an embedding of the category of Δ-filtered $\mathcal{A}(\tilde{\mathbf{a}})$-modules in to the category of Δ-filtered $\mathcal{A}(\mathbf{a})$-modules. The correspondence alluded to above then implies that for $\tilde{\mathbf{a}} \leq \mathbf{a}$, if there are finitely $P(\mathbf{d})$-orbits in $\mathfrak{q}_u(\mathbf{a}, \mathbf{d})$ for all $\mathbf{d}$, then there are finitely many $P(\tilde{\mathbf{d}})$-orbits in $\mathfrak{q}_u(\tilde{\mathbf{a}}, \tilde{\mathbf{d}})$ for all $\tilde{\mathbf{d}}$.

Remark 1.3. Using the results from [4] and the Auslander–Reiten quivers that we have calculated, it is possible to calculate the degenerations of $P(\mathbf{d})$-orbits in $\mathfrak{q}_u(\mathbf{a}, \mathbf{d})$ in the finite cases. More precisely, let $\mathcal{O}, \mathcal{O}'$ be P-orbits and M, M' the corresponding $\mathcal{A}(\mathbf{a})$-modules. In [4] it is shown that M is greater than M' in the *hom-order* if and only if $\mathcal{O}$ is a degeneration of $\mathcal{O}'$. The hom-order can be read off from the Auslander–Reiten quiver.

2. Proof of Theorem 1.1

Let $\mathbf{a} \in \mathbb{Z}_{\geq 1}^3$ and $\mathbf{d} \in \mathbb{Z}_{\geq 1}^t$ with $a_1 + a_2 + a_3 = t$. Let $P = P(\mathbf{d})$ and $Q = Q(\mathbf{a}, \mathbf{d})$. Write $Q_u = Q_u(\mathbf{a}, \mathbf{d})$ for the unipotent radical of Q and $\mathfrak{q}'_u = \mathfrak{q}'_u(\mathbf{a}, \mathbf{d})$ for the derived subalgebra of $\mathfrak{q}_u = \mathfrak{q}_u(\mathbf{a}, \mathbf{d})$. Define $b_i = \sum_{j=1}^{i} a_j$ for $i = 0, 1, 2, 3$ as in the introduction.

Proof of (1). The action of P on $\mathfrak{q}_u$ induces an action of P/Q_u on $\mathfrak{q}_u/\mathfrak{q}'_u$, and if P acts on $\mathfrak{q}_u$ with finitely many orbits, then there is necessarily a finite number of P/Q_u-orbits in $\mathfrak{q}_u/\mathfrak{q}'_u$. For the dimension vectors given in the second column of Table 1, we have $\dim P/Q_u = \dim \mathfrak{q}_u/\mathfrak{q}'_u$; this dimension is given in the third column. Observe that the scalar matrices in P/Q_u act trivially on $\mathfrak{q}_u/\mathfrak{q}'_u$. Therefore, there cannot be a dense P/Q_u-orbit on $\mathfrak{q}_u/\mathfrak{q}'_u$ and hence there must be infinitely many P/Q_u-orbits. This proves part (1) of Theorem 1.1.

Remark 2.1. The above proof of (1) is obtained by considering a particluar quadratic form in the dimension vectors $\mathbf{d} \in \mathbb{Z}_{\geq 1}^t$. For a fixed $\mathbf{a} = (a_1, a_2, a_3)$ with $a_1 + a_2 + a_3 = t$ and b_i $(0 \leq i \leq 3)$ as before, set $I = \{(i, j) \mid b_{k-1} < i < j \leq b_k \text{ for some } k = 1, 2, 3\}$ and $J = \{(i, j) \mid b_{k-1} < i \leq b_k < j \leq b_{k+1} \text{ for some } k = 1, 2\}$. The quadratic form at $\mathbf{d} \in \mathbb{Z}_{\geq 1}^t$ is then given by

$$\sum_{i=1}^{t} d_i^2 + \sum_{(i,j) \in I} d_i d_j - \sum_{(i,j) \in J} d_i d_j.$$

On readily checks that, by our construction, this expression equals $\dim P/Q_u - \dim \mathfrak{q}_u/\mathfrak{q}'_u$.

For each of the triples $\mathbf{a}$ in Table 1, this quadratic form is semi-definite and the value of $\mathbf{d}$ given in the table is the unique indivisible vector in the kernel of the quadratic form. For a classification of critical quadratic unit forms we refer to [10] and for further applications in representation theory to Ringel's monograph [14].

Proof of (2). In order to prove part (2) of Theorem 1.1, we translate the problem in to one regarding the representation theory of a certain quasi-hereditary algebra $\mathcal{A}(\mathbf{a})$. The algebra $\mathcal{A}(\mathbf{a})$ is defined as the quotient of a path algebra by some relations as explained below.

The quiver $\mathcal{Q} = \mathcal{Q}(\mathbf{a})$ is defined to have vertex set $\{1, \ldots, t\}$; there are arrows $\alpha_i : i \to i+1$ for all i, an arrow $\beta_{b_1} : b_2 \to b_1$ and an arrow $\beta_{b_2} : b_3 \to b_2$. Below, in Figure 1, we give an example of a quiver $\mathcal{Q}(\mathbf{a})$.
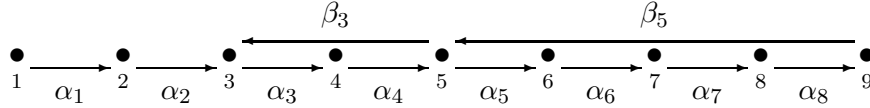

FIGURE 1. The quiver $\mathcal{Q}(3, 2, 4)$

Let I be the ideal of the path algebra $k\mathcal{Q}$ of $\mathcal{Q}$ generated by the relations:

$$(2.1) \qquad \beta_{b_1}\alpha_{b_2-1}\cdots\alpha_{b_1+1}\alpha_{b_1} = 0 \quad \text{and} \quad \alpha_{b_2-1}\cdots\alpha_{b_1+1}\alpha_{b_1}\beta_{b_1} = \beta_{b_2}\alpha_{b_3-1}\cdots\alpha_{b_2+1}\alpha_{b_2}.$$

The algebra $\mathcal{A} = \mathcal{A}(\mathbf{a})$ is defined to be the quotient $k\mathcal{Q}/I$.

Recall that an $\mathcal{A}$-module is given by a family of vector spaces M_i for $i = 1, \ldots, t$ and linear maps $M_{\alpha_i} : M_i \to M_{i+1}$ for $i = 1, \ldots, t-1$, and $M_{\beta_{b_i}} : M_{b_{i+1}} \to M_{b_i}$ for $i = 1, 2$ that satisfy the relations (2.1). For an arrow γ in $\mathcal{Q}$ we often simply write γ rather than M_γ when considering an $\mathcal{A}$-module M.

Note that the arguments of [2, §2–3] apply in the present situation showing that $\mathcal{A}$ is a quasi-hereditary algebra, and that the isoclasses of Δ-filtered $\mathcal{A}$-modules with Δ-dimension vector $\mathbf{d}$ are in bijective correspondence with the adjoint orbits of $P = P(\mathbf{d})$ on $\mathfrak{q}_u = \mathfrak{q}_u(\mathbf{a}, \mathbf{d})$.

Define the category $\mathcal{M}(\mathbf{a})$ to be the full subcategory of $\mathcal{A}(\mathbf{a})$-mod consisting of modules M for which M_{α_i} is injective for all $i = 1, \ldots, t-1$. Let $\mathcal{M}(\mathbf{a}, \mathbf{d})$ be the class of modules M in $\mathcal{M}(\mathbf{a})$ for which $\dim M_i = e_i = \sum_{j=1}^{i} d_j$ for all $i = 1, \ldots, t$. Our next result can be proved in exactly the same way as [2, Lem. 1 and Lem. 2].

Lemma 2.2. *The adjoint orbits of $P(\mathbf{d})$ on $\mathfrak{q}_u(\mathbf{a}, \mathbf{d})$ are in bijective correspondence with the isoclasses of modules in $\mathcal{M}(\mathbf{a}, \mathbf{d})$.*

Define the *standard $\mathcal{A}(\mathbf{a})$-module* $\Delta(i)$ as follows. For $j < i$, we have $\Delta(i)_j = 0$ and for $j \geq i$, we have $\Delta(i)_j = k$. The arrows α_j act as the zero map for $j < i$ and as the identity map for $j \geq i$; the arrows β_{b_1} and β_{b_2} both act as zero maps. With these definitions one can prove the following lemma in exactly the same way as [2, Prop. 1].

Lemma 2.3. *The algebra $\mathcal{A}(\mathbf{a})$ is quasi-hereditary with the inverse order on the index set $\{1, \ldots, t\}$ and standard modules as defined above. The category $\mathcal{M}(\mathbf{a})$ is precisely the category $\mathcal{F}(\mathcal{A}(\mathbf{a}), \Delta)$ of Δ-filtered $\mathcal{A}(\mathbf{a})$-modules.*

Recall that an $\mathcal{A}$-module M is called Δ-*filtered* if there exists a filtration $0 = M_0 \subseteq M_1 \subseteq \ldots \subseteq M_r = M$ of M, where for each i we have $M_i/M_{i-1} \cong \Delta(j)$ for some j; such a filtration of M is called a Δ-*filtration*. Given a Δ-filtered module M, define the Δ-*dimension vector* $\mathbf{dim}_\Delta M \in \mathbb{Z}_{\geq 0}^t$ of M by setting $(\mathbf{dim}_\Delta M)_j$ equal to the number of factors isomorphic to $\Delta(j)$

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in a Δ-filtration of M; it is a standard result that the Δ-dimension vector is well-defined. Recall that a quasi-hereditary algebra is said to have *finite Δ-representation type* if there are only finitely many isoclasses of indecomposable Δ-filtered modules. For general results on categories of Δ-filtered modules over quasi-hereditary algebras, we refer to [5].

It follows from Lemmas 2.2 and 2.3 that the $P(\mathbf{d})$-orbits in $\mathfrak{q}_u(\mathbf{a}, \mathbf{d})$ are in bijective correspondence with the Δ-filtered $\mathcal{A}(\mathbf{a})$-modules with Δ-dimension vector $\mathbf{d}$. Using this equivalence we verify part (2) of Theorem 1.1 by showing that $\mathcal{A}(\mathbf{a})$ has finite Δ-representation type for each triple $\mathbf{a}$ in Table 2. We achieve this by calculating the Auslander–Reiten quiver of $\mathcal{F}(\mathcal{A}(\mathbf{a}), \Delta)$, for each such $\mathbf{a}$. For the values of $\mathbf{a}$ involving a parameter, we calculate the Auslander–Reiten quiver for a particular value of that parameter; it is then straightforward to generalize to the generic case. For completeness, the Auslander–Reiten quivers are given in the appendix; next we give a sketch of their construction.

Thanks to [15], the category of Δ-filtered modules for a quasi-hereditary algebra has almost split sequences. Therefore, the Auslander–Reiten quiver is defined for $\mathcal{F}(\mathcal{A}(\mathbf{a}), \Delta)$. To calculate these quivers, we use standard methods as explained in [1, Ch. VII], see also [14]. We begin by taking a $\mathbb{Z}$-covering of the algebra $\mathcal{A}(\mathbf{a})$ and then making a "cut"; we denote the resulting algebra by $\mathcal{A}_{\mathbb{Z}}(\mathbf{a})$. The category $\mathcal{F}(\mathcal{A}_{\mathbb{Z}}(\mathbf{a}), \Delta)$ admits a simple projective object. Thus we can construct the AR-quiver for $\mathcal{F}(\mathcal{A}_{\mathbb{Z}}(\mathbf{a}), \Delta)$ by "knitting". After several steps repetitions occur and as a result, one can obtain the entire AR-quiver for $\mathcal{F}(\mathcal{A}(\mathbf{a}), \Delta)$.

Proof of (3). To verify the first assertion of (3), it suffices to check that if we obtain the triple $\tilde{\mathbf{a}}$ from a triple $\mathbf{a}$ in Table 1 by subtracting 1 from an entry of $\mathbf{a}$, then $\tilde{\mathbf{a}}$ is less than or equal to one of the triples in Table 2. This is an elementary case by case check and is left to the reader.

Let $\tilde{\mathbf{a}} \leq \mathbf{a}$. We show below that there is embedding of $\mathcal{F}(\mathcal{A}(\tilde{\mathbf{a}}), \Delta)$ into $\mathcal{F}(\mathcal{A}(\mathbf{a}), \Delta)$. Along with the correspondence between $P(\mathbf{d})$-orbits in $\mathfrak{q}_u(\mathbf{a}, \mathbf{d})$ and Δ-filtered $\mathcal{A}(\mathbf{a})$-modules with Δ-dimension vector $\mathbf{d}$ this implies the second assertion of (3). The embedding identifies $\mathcal{F}(\mathcal{A}(\tilde{\mathbf{a}}), \Delta)$ with the subcategory of $\mathcal{F}(\mathcal{A}(\mathbf{a}), \Delta)$ consisting of modules with Δ-dimension vector $\mathbf{d}$ such that $d_i = 0$ for $i = \tilde{a}_1+1, \ldots, b_1$, $i = b_1+\tilde{a}_2+1, \ldots, b_2$ and $i = b_2+\tilde{a}_3+1, \ldots, b_3$.

In order to define the embedding, we require some notation. Define $\tilde{b}_i$ for $i = 0, 1, 2, 3$ in analogy to the definition of b_i. The map $f : \{1, \ldots, \tilde{b}_3\} \to \{1, \ldots, b_3\}$ is defined by setting:

$$f(i) = \begin{cases} i - b_j + \tilde{b}_j & \text{if} & b_j & < i \leq & b_j + \tilde{a}_{j+1}; \\ \tilde{b}_{j+1} & \text{if} & b_j + \tilde{a}_{j+1} & < i \leq & b_{j+1}. \end{cases}$$

Let $\tilde{M}$ be a Δ-filtered $\mathcal{A}(\tilde{\mathbf{a}})$-module. We obtain a Δ-filtered $\mathcal{A}(\mathbf{a})$-module M from $\tilde{M}$ as follows. Set $M_i = \tilde{M}_{f(i)}$ for $i = 1, \ldots, t$. If $f(i+1) = f(i)$, then define M_{α_i} to be the identity map; and if $f(i+1) = f(i) + 1$, then define $M_{\alpha_i} = \tilde{M}_{\alpha_{f(i)}}$. Finally, define, $M_{\beta_{b_i}} = \tilde{M}_{\beta_{\tilde{b}_i}}$ for $i = 1, 2$. This construction defines a functor from $\mathcal{F}(\mathcal{A}(\tilde{\mathbf{a}}), \Delta)$ to $\mathcal{F}(\mathcal{A}(\mathbf{a}), \Delta)$ by $\tilde{M} \mapsto M$. It is straightforward to check that this an embedding. This completes the proof of (3).

Appendix A. Auslander–Reiten quivers

In Figures 2–7 we include the Auslander–Reiten quivers of the Δ-filtered modules for the algebras $\mathcal{A}(\mathbf{a})$ in case of finite Δ-representation type. The $\mathcal{A}(\mathbf{a})$-modules are denoted by giving their filtration into standard modules. The right and left boundary of the Auslander–Reiten quivers must be identified, it is a cylinder or a Möbius band.
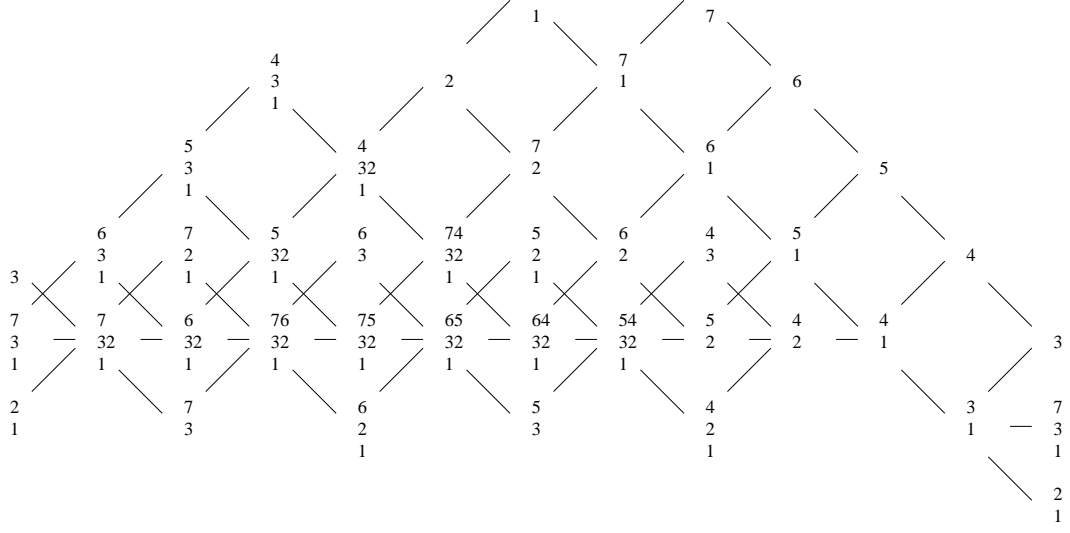
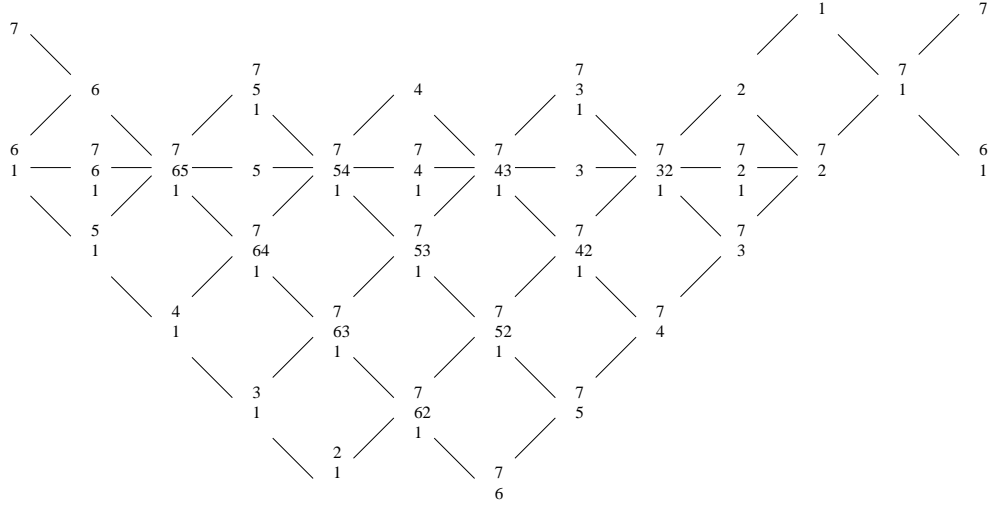FIGURE 2. The AR-quiver of Δ-filtered modules for $\mathcal{A}(1,2,a)$ for $a=4$

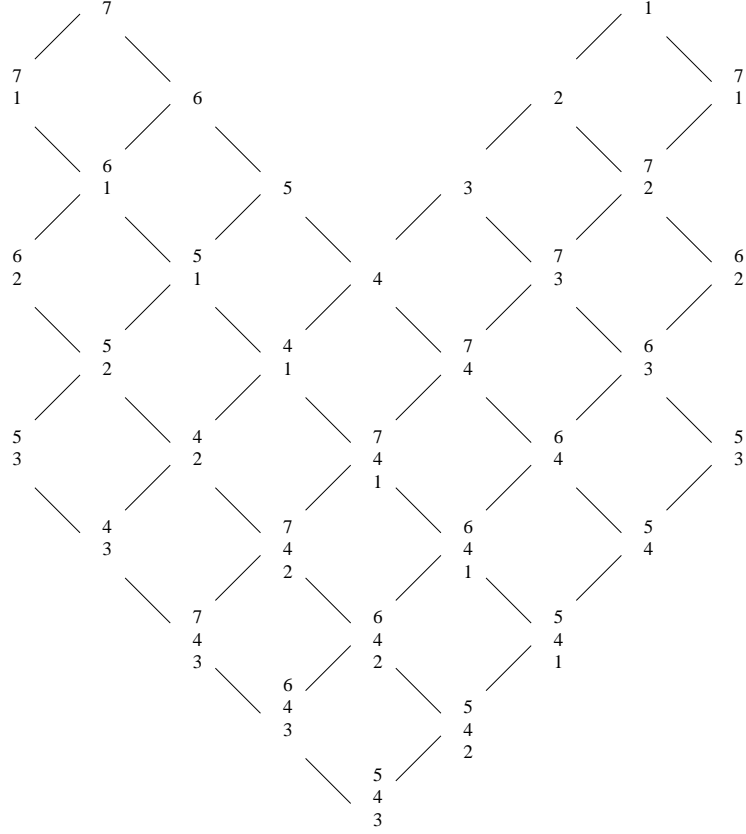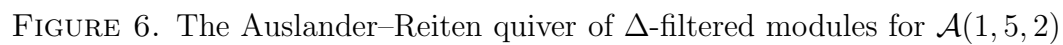FIGURE 3. The AR-quiver of Δ-filtered modules for $\mathcal{A}(1,a,1)$ for $a=5$
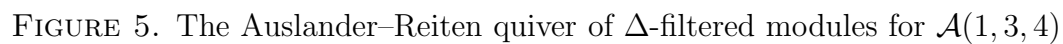
FIGURE 4. The AR-quiver of Δ-filtered modules for $\mathcal{A}(a, 1, b)$ for $a = 3$, $b = 3$
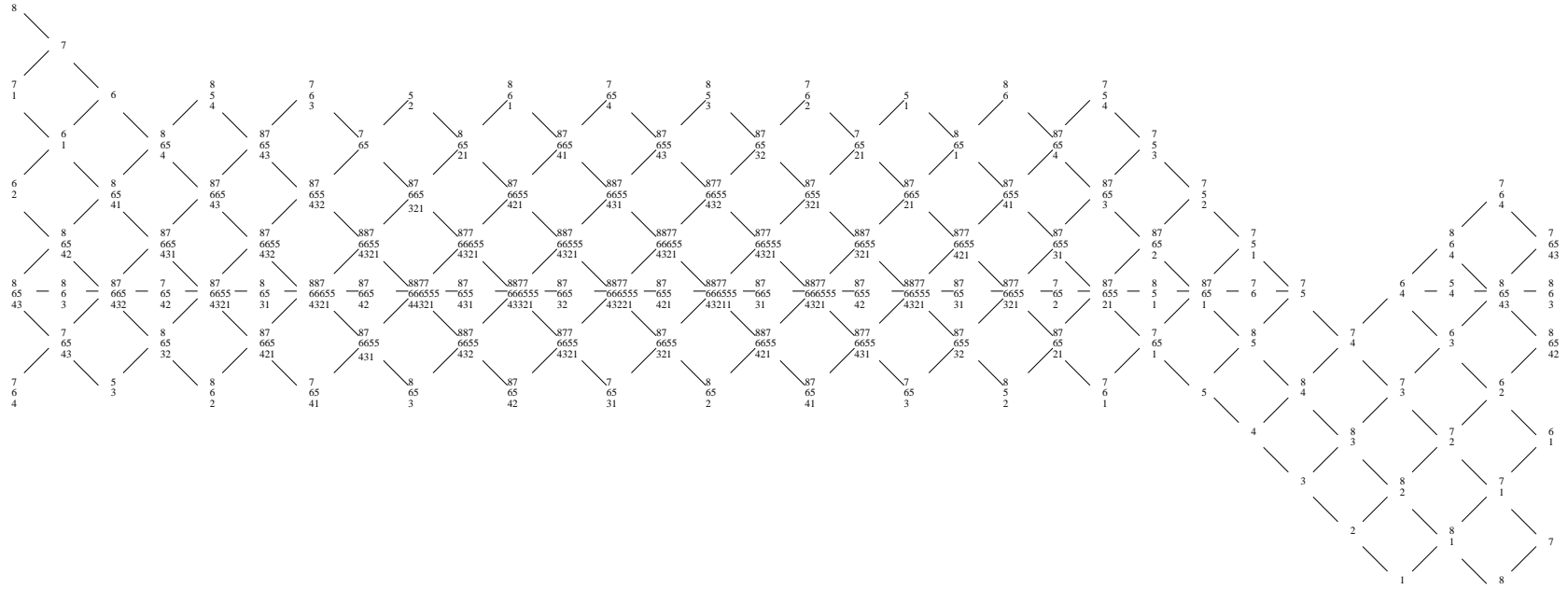
FIGURE 5. The Auslander–Reiten quiver of Δ-filtered modules for $\mathcal{A}(1,3,4)$

FIGURE 6. The Auslander–Reiten quiver of Δ-filtered modules for $\mathcal{A}(1,5,2)$

FIGURE 7. The Auslander–Reiten quiver of Δ-filtered modules for $\mathcal{A}(4,2,2)$

Acknowledgements

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